MASS, MOMENTUM, AND ENERGY EQUATIONS
This chapter deals with four equations commonly used in fluid mechanics: the mass, Bernoulli, Momentum and energy equations.

• The *mass equation* is an expression of the conservation of mass principle.

• The *Bernoulli equation* is concerned with the conservation of kinetic, potential, and flow energies of a fluid stream and their conversion to each other in regions of flow where net viscous forces are negligible and where other restrictive conditions apply. The *energy equation* is a statement of the conservation of energy principle.

• In fluid mechanics, it is found convenient to separate *mechanical energy* from *thermal energy* and to consider the conversion of mechanical energy to thermal energy as a result of frictional effects as *mechanical energy loss*. Then the energy equation becomes the *mechanical energy balance*. 
We start this chapter with an overview of conservation principles and the conservation of mass relation. This is followed by a discussion of various forms of mechanical energy. Then we derive the Bernoulli equation by applying Newton’s second law to a fluid element along a streamline and demonstrate its use in a variety of applications. We continue with the development of the energy equation in a form suitable for use in fluid mechanics and introduce the concept of head loss. Finally, we apply the energy equation to various engineering systems.
Conservation of Mass

- The conservation of mass relation for a closed system undergoing a change is expressed as $m_{sys} = \text{constant}$ or $dm_{sys}/dt = 0$, which is a statement of the obvious that the mass of the system remains constant during a process.

- For a control volume (CV) or open system, mass balance is expressed in the rate form as

  $$\dot{m}_{in} - \dot{m}_{out} = \frac{dm_{CV}}{dt}$$

  where $\dot{m}_{in}$ and $\dot{m}_{out}$ are the total rates of mass flow into and out of the control volume, respectively, and $dm_{CV}/dt$ is the rate of change of mass within the control volume boundaries.

- In fluid mechanics, the conservation of mass relation written for a differential control volume is usually called the continuity equation.
The conservation of mass principle for a control volume can be expressed as: The net mass transfer to or from a control volume during a time interval $t$ is equal to the net change (increase or decrease) in the total mass within the control volume during $t$. That is,

\[
\left( \text{Total mass entering the CV during } \Delta t \right) - \left( \text{Total mass leaving the CV during } \Delta t \right) = \left( \text{Net change in mass within the CV during } \Delta t \right)
\]

or

\[
m_{\text{in}} - m_{\text{out}} = \Delta m_{\text{CV}} \quad (\text{kg})
\]
where \( \Delta m_{CV} = m_{\text{final}} - m_{\text{initial}} \) is the change in the mass of the control volume during the process. It can also be expressed in rate form as

\[
in_{\text{in}} - in_{\text{out}} = dm_{CV}/dt \quad \text{(kg/s)}
\]

The Equations above are often referred to as the **mass balance** and are applicable to any control volume undergoing any kind of process.
Mass Balance for Steady-Flow Processes

- During a steady-flow process, the total amount of mass contained within a control volume does not change with time ($m_{CV} = \text{constant}$). Then the conservation of mass principle requires that the total amount of mass entering a control volume equal the total amount of mass leaving it.

$$\sum m_{\text{in}} = \sum m_{\text{out}} \quad \text{(kg/s)}$$

When dealing with steady-flow processes, we are not interested in the amount of mass that flows in or out of a device over time; instead, we are interested in the amount of mass flowing per unit time, that is, the **mass flow rate** $\dot{m}$.

It states that **the total rate of mass entering a control volume is equal to the total rate of mass leaving it**.
Many engineering devices such as nozzles, diffusers, turbines, compressors, and pumps involve a single stream (only one inlet and one outlet).

For these cases, we denote the inlet state by the subscript 1 and the outlet state by the subscript 2, and drop the summation signs

\[
\text{Steady flow (single stream):} \quad m_1 = m_2 \quad \rightarrow \quad \rho_1 V_1 A_1 = \rho_2 V_2 A_2
\]
EXAMPLE 2–1 : Water Flow through a Garden Hose Nozzle
A garden hose attached with a nozzle is used to fill a 10-gal bucket. The inner diameter of the hose is 2 cm, and it reduces to 0.8 cm at the nozzle exit (Fig. 5–12). If it takes 50 s to fill the bucket with water, determine (a) the volume and mass flow rates of water through the hose, and (b) the average velocity of water at the nozzle exit.

Assumptions 1 Water is an incompressible substance. 2 Flow through the hose is steady. 3 There is no waste of water by splashing.

Properties We take the density of water to be 1000 kg/m3  1 kg/L.

Analysis (a) Noting that 10 gal of water are discharged in 50 s, the volume and mass flow rates of water are

\[
\begin{align*}
\dot{V} &= \frac{V}{\Delta t} = \frac{10 \text{ gal}}{50 \text{ s}} \left(\frac{3.7854 \text{ L}}{1 \text{ gal}}\right) = 0.757 \text{ L/s} \\
\dot{m} &= \rho \dot{V} = (1 \text{ kg/L})(0.757 \text{ L/s}) = 0.757 \text{ kg/s}
\end{align*}
\]
(b) The cross-sectional area of the nozzle exit is

\[ A_e = \pi r_e^2 = \pi (0.4 \text{ cm})^2 = 0.5027 \text{ cm}^2 = 0.5027 \times 10^{-4} \text{ m}^2 \]

The volume flow rate through the hose and the nozzle is constant. Then the average velocity of water at the nozzle exit becomes

\[ V_e = \frac{\dot{V}}{A_e} = \frac{0.757 \text{ L/s}}{0.5027 \times 10^{-4} \text{ m}^2} \left( \frac{1 \text{ m}^3}{1000 \text{ L}} \right) = 15.1 \text{ m/s} \]
Many fluid systems are designed to transport a fluid from one location to another at a specified flow rate, velocity, and elevation difference, and the system may generate mechanical work in a turbine or it may consume mechanical work in a pump or fan during this process.

These systems do not involve the conversion of nuclear, chemical, or thermal energy to mechanical energy. Also, they do not involve any heat transfer in any significant amount, and they operate essentially at constant temperature.

Such systems can be analyzed conveniently by considering the mechanical forms of energy only and the frictional effects that cause the mechanical energy to be lost (i.e., to be converted to thermal energy that usually cannot be used for any useful purpose).

The mechanical energy can be defined as the form of energy that can be converted to mechanical work completely and directly by an ideal mechanical device.

Kinetic and potential energies are the familiar forms of mechanical energy.
Therefore, the mechanical energy of a flowing fluid can be expressed on a unit-mass basis as

\[ e_{\text{mech}} = \frac{P}{\rho} + \frac{V^2}{2} + gz \]

In the absence of any changes in flow velocity and elevation, the power produced by an ideal hydraulic turbine is proportional to the pressure drop of water across the turbine.
Most processes encountered in practice involve only certain forms of energy, and in such cases it is more convenient to work with the simplified versions of the energy balance. For systems that involve only *mechanical forms of energy* and its transfer as *shaft work*, the conservation of energy principle can be expressed conveniently as

\[
E_{\text{mech, in}} - E_{\text{mech, out}} = \Delta E_{\text{mech, system}} + E_{\text{mech, loss}}
\]

where \( E_{\text{mech, loss}} \) represents the conversion of mechanical energy to thermal energy due to irreversibilities such as friction. For a system in steady operation, the mechanical energy balance becomes

\[
E_{\text{mech, in}} = E_{\text{mech, out}} + E_{\text{mech, loss}}
\]
The Bernoulli equation is an approximate relation between pressure, velocity, and elevation, and is valid in regions of steady, incompressible flow where net frictional forces are negligible (as shown in the Figure below). Despite its simplicity, it has proven to be a very powerful tool in fluid mechanics.

The Bernoulli equation is an approximate equation that is valid only in in viscid regions of flow where net viscous forces are negligibly small compared to inertial, gravitational, or pressure forces. Such regions occur outside of boundary layers and wakes.
Derivation of the Bernoulli Equation

the Bernoulli Equation is derived from the mechanical energy equation

\[ \Delta e_{	ext{mech}} = \frac{P_2 - P_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1) \quad (\text{kJ/kg}) \]

since the we are dealing with steady flow system without the effect of the mechanical work and the friction on the system the first terms become zero.

**Steady, incompressible flow:** \( \frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant (along a streamline)} \)

This is the famous **Bernoulli equation**, which is commonly used in fluid mechanics for steady, incompressible flow along a streamline in inviscid regions of flow.
The Bernoulli equation can also be written between any two points on the same streamline as

\[
\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2
\]

*Steady, incompressible flow:*
Limitations on the Use of the Bernoulli Equation

- **Steady flow** The first limitation on the Bernoulli equation is that it is applicable to *steady flow*.
- **Frictionless flow** Every flow involves some friction, no matter how small, and *frictional effects* may or may not be negligible.
- **No shaft work** The Bernoulli equation was derived from a force balance on a particle moving along a streamline.
- **Incompressible flow** One of the assumptions used in the derivation of the Bernoulli equation is that \( \rho = \) constant and thus the flow is incompressible.
- **No heat transfer** The density of a gas is inversely proportional to temperature, and thus the Bernoulli equation should not be used for flow sections that involve significant temperature change such as heating or cooling sections.
- Strictly speaking, the Bernoulli equation \( P\rho + \frac{V^2}{2} + gz = C \) is applicable along a streamline, and the value of the constant \( C \), in general, is different for different streamlines. But when a region of the flow is *irrotational*, and thus there is no *vorticity* in the flow field, the value of the constant \( C \) remains the same for all streamlines, and, therefore, the Bernoulli equation becomes applicable *across* streamlines as well.
\[ \frac{P_1}{\rho} + \frac{V_1^2}{2} + g\zeta_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + g\zeta_2 \]

**FIGURE 5–32**

When the flow is irrotational, the Bernoulli equation becomes applicable between any two points along the flow (not just on the same streamline).
EXAMPLE 2–2 Spraying Water into the Air
Water is flowing from a hose attached to a water main at 400 kPa gage (Fig. below). A child places his thumb to cover most of the hose outlet, causing a thin jet of high-speed water to emerge. If the hose is held upward, what is the maximum height that the jet could achieve?

This problem involves the conversion of flow, kinetic, and potential energies to each other without involving any pumps, turbines, and wasteful components with large frictional losses, and thus it is suitable for the use of the Bernoulli equation. The water height will be maximum under the stated assumptions. The velocity inside the hose is relatively low \( (V_1 = 0) \) and we take the hose outlet as the reference level \( (z_1 = 0) \). At the top of the water trajectory \( V_2 = 0 \), and atmospheric pressure pertains. Then the Bernoulli equation simplifies to
\[
\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2
\]

Solving for \(z_2\) and substituting,

\[
z_2 = \frac{P_1 - P_{\text{atm}}}{\rho g} = \frac{P_{1,\ gage}}{\rho g} = \frac{400 \text{ kPa}}{(1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)} \left( \frac{1000 \text{ N/m}^2}{1 \text{ kPa}} \right) \left( \frac{1 \text{ kg \cdot m/s}^2}{1 \text{ N}} \right)
\]

\[= 40.8 \text{ m}\]
EXAMPLE 2-3 Water Discharge from a Large Tank

A large tank open to the atmosphere is filled with water to a height of 5 m from the outlet tap (Fig. below). A tap near the bottom of the tank is now opened, and water flows out from the smooth and rounded outlet. Determine the water velocity at the outlet.

This problem involves the conversion of flow, kinetic, and potential energies to each other without involving any pumps, turbines, and wasteful components with large frictional losses, and thus it is suitable for the use of the Bernoulli equation. We take point 1 to be at the free surface of water so that $P_1 = P_{atm}$ (open to the atmosphere), $V_1 = 0$ (the tank is large relative to the outlet), and $z_1 = 5$ m and $z_2 = 0$ (we take the reference level at the center of the outlet). Also, $P_2 = P_{atm}$ (water discharges into the atmosphere).
Then the Bernoulli equation simplifies to

\[
\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 \quad \rightarrow \quad z_1 = \frac{V_2^2}{2g}
\]

Solving for \( V_2 \) and substituting

\[
V_2 = \sqrt{2gz_1} = \sqrt{2(9.81 \text{ m/s}^2)(5 \text{ m})} = 9.9 \text{ m/s}
\]

The relation \( V = \sqrt{2gz} \) is called the Toricelli equation.
EXAMPLE 2–4 Siphoning Out Gasoline from a Fuel Tank
During a trip to the beach ($P_{atm} = 1 \text{ atm} = 101.3 \text{ kPa}$), a car runs out of gasoline, and it becomes necessary to siphon gas out of the car of a Good Samaritan (Fig. below). The siphon is a small-diameter hose, and to start the siphon it is necessary to insert one siphon end in the full gas tank, fill the hose with gasoline via suction, and then place the other end in a gas can below the level of the gas tank. The difference in pressure between point 1 (at the free surface of the gasoline in the tank) and point 2 (at the outlet of the tube) causes the liquid to flow from the higher to the lower elevation. Point 2 is located 0.75 m below point 1 in this case, and point 3 is located 2 m above point 1. The siphon diameter is 4 mm, and frictional losses in the siphon are to be disregarded. Determine (a) the minimum time to withdraw 4 L of gasoline from the tank to the can and (b) the pressure at point 3. The density of gasoline is 750 kg/m$^3$. 

![Diagram of siphoning system]
**Analysis** (a) We take point 1 to be at the free surface of gasoline in the tank so that $P_1 = P_{\text{atm}}$ (open to the atmosphere), $V_1 = 0$ (the tank is large relative to the tube diameter), and $z_2 = 0$ (point 2 is taken as the reference level). Also, $P_2 = P_{\text{atm}}$ (gasoline discharges into the atmosphere). Then the Bernoulli equation simplifies to

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

Solving for $V_2$ and substituting,

$$V_2 = \sqrt{2gz_1} = \sqrt{2(9.81 \text{ m/s}^2)(0.75 \text{ m})} = 3.84 \text{ m/s}$$

The cross-sectional area of the tube and the flow rate of gasoline are

$$A = \pi D^2/4 = \pi (5 \times 10^{-3} \text{ m})^2/4 = 1.96 \times 10^{-5} \text{ m}^2$$

$$\dot{V} = V_2 A = (3.84 \text{ m/s})(1.96 \times 10^{-5} \text{ m}^2) = 7.53 \times 10^{-5} \text{ m}^3/\text{s} = 0.0753 \text{ L/s}$$

Then the time needed to siphon 4 L of gasoline becomes

$$\Delta t = \frac{\dot{V}}{\dot{V}} = \frac{4 \text{ L}}{0.0753 \text{ L/s}} = 53.1 \text{ s}$$
(b) The pressure at point 3 can be determined by writing the Bernoulli equation between points 2 and 3. Noting that \( V_2 = V_3 \) (conservation of mass), \( z_2 = 0 \), and \( P_2 = P_{\text{atm}} \),

\[
\frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 = \frac{P_3}{\rho g} + \frac{V_3^2}{2g} + z_3
\]

\[
\rightarrow \quad \frac{P_{\text{atm}}}{\rho g} = \frac{P_3}{\rho g} + z_3
\]

Solving for \( P_3 \) and substituting,

\[
P_3 = P_{\text{atm}} - \rho gz_3
\]

\[
= 101.3 \text{ kPa} - (750 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(2.75 \text{ m})\left(\frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m/s}^2}\right)\left(\frac{1 \text{ kPa}}{1000 \text{ N/m}^2}\right)
\]

\[
= 81.1 \text{ kPa}
\]